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Optimality of the boundary control for $n \times n$ parabolic lag system

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Abstract

Optimal boundary control problem for $n \times n$ coupled system of second order parabolic lag partial differential equation with infinitely many variables is considered. By Lions scheme [J.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Grundlehren Math. Wiss., vol. 170, Springer-Verlag, 1971], necessary and sufficient condition of optimality for the Neumann problem with quadratic performance functional and constrained control is derived. Finally, several mathematical examples for derived optimality conditions are presented.

Keywords: $n \times n$ parabolic system; Neumann problem; Second order operator; Infinite number of variables; Boundary control problem; Optimality condition

0. Introduction

The linear quadratic optimal control problem described by distributed parameter system has a variety of mechanical and technical sources and applications. Fundamental class of optimal controls and its mathematical approaches can be found in Lions [17].

The necessary and sufficient conditions of optimality for systems ($n \times n$ systems) governed by different types of partial differential operators defined on spaces of functions of infinitely many

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variables are discussed in [2–12]. The interest in the study of this class of operators is stimulated by problems in quantum field theory [1].

Various optimization problems associated with the optimal control of distributed parameter systems with time-varying lags have been studied by Kowalewski in [13–16].

In this paper, we consider optimal boundary control problem for $n \times n$ second order parabolic partial differential coupled system with infinitely many variables, in which time-varying lag appears in the equation and in the Neumann boundary condition simultaneously. Such systems constitute in a linear approximation a universal mathematical model for many processes of optimal heating.

Necessary and sufficient condition of optimality with a quadratic performance functional and constrained control is derived for the Neumann problem. Then we find the set of inequalities which characterize an optimal boundary control. This set is studied in order to construct computations for the approximation of the control. Finally, several mathematical examples for derived optimality conditions are presented.

The plan of this paper is the following. In Section 1, we formulate the mixed Neumann problem for $n \times n$ second order parabolic coupled system with infinitely many variables and time-varying lag. In Section 2, we study the linear quadratic boundary control problem then we find the set of inequalities defining the necessary and sufficient condition for optimality of the boundary control. In Section 3, we give several mathematical examples for derived optimality conditions.

1. Mixed Neumann problem for $n \times n$ parabolic lag system

Below we consider the functions of points $x \in R^\infty = R^1 \times R^1 \times \cdots$, the coordinate notation of such points is $x = (x_k)_{k=1}^\infty$, $x_k \in R^1$. Let $(P_k(x_k))_{k=1}^\infty$ be a fixed sequence of continuous positive probability weights, i.e., $\int_{R^1} P_k(x_k)\,dx = 1$. The measure on R^∞ given by

$$d\rho(x) = \big(P_1(x_1)\,dx_1\big) \otimes \big(P_2(x_2)\,dx_2\big) \otimes \cdots = \big(d\rho_1(x_1)\big) \otimes \big(d\rho_2(x_2)\big) \otimes \cdots$$

is called a (weighted) product measure [1].

Let Ω be a bounded open domain in R^∞ with smooth boundary Γ and denote by $(W^1(\Omega, R^\infty))^n$ n Cartesian product of the Sobolev spaces of vector function $\bar{y}(x) = \bar{y} = (y_1, y_2, \ldots, y_n) = (y_i)_{i=1}^n$ defined on Ω [8,9].

For each variable t which denotes the time, $t \in (0, T)$, $T < \infty$, we define a family of bilinear functionals on $(W^1(\Omega, R^\infty))^n$ by

$$\pi : \big(W^1\big(\Omega, R^\infty\big)\big)^n \times \big(W^1\big(\Omega, R^\infty\big)\big)^n \to R^1,$$
$$\pi(t; \bar{y}, \bar{\varphi}) = \big(A(t)\bar{y}, \bar{\varphi}\big)_{(L_2(\Omega, R^\infty))^n}, \quad \bar{y} = (y_i)_{i=1}^n,\ \bar{\varphi} = (\varphi_i)_{i=1}^n \in \big(W^1\big(\Omega, R^\infty\big)\big)^n, \tag{1}$$

where $A(t)$ is $n \times n$ matrix operator which maps $(W^1(\Omega, R^\infty))^n$ onto $(W^{-1}(\Omega, R^\infty))^n$ and takes the form

$$A(t) = \begin{pmatrix} -\sum_{k=1}^\infty D_k^2 + q + 1 & -1 & \cdots & -1 \\ 1 & -\sum_{k=1}^\infty D_k^2 + q + 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -\sum_{k=1}^\infty D_k^2 + q + 1 \end{pmatrix}_{n \times n}$$

that is

$$A(t)y_i(x) = \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t)\right) y_i(x) + \sum_{j=1}^{n} a_{ij} y_j(x)$$
$$= -\sum_{k=1}^{\infty} \frac{1}{\sqrt{P_k(x_k,t)}} \frac{\partial^2}{\partial x_k^2} \sqrt{P_k(x_k,t)}\, y_i(x) + q(x,t) y_i(x) + \sum_{j=1}^{n} a_{ij} y_j(x),$$

where $(-\sum_{k=1}^{\infty} D_k^2 + q(x,t))$ is a bounded second order self-adjoint elliptic partial differential operator with an infinite number of variables [1],

$$D_k y_i(x) = \frac{1}{\sqrt{P_k(x_k,t)}} \frac{\partial}{\partial x_k} \sqrt{P_k(x_k,t)}\, y_i(x),$$

$q(x,t)$ is a real-valued function in Ω which is bounded and measurable on Ω, such that $q(x,t) \geqslant \nu > 1$, ν is a constant and $\{a_{ij}\}$ is the coefficient matrix such that

$$a_{ij} = \begin{cases} 1 & \text{if } i \geqslant j, \\ -1 & \text{if } i < j. \end{cases}$$

In [9], we proved that the above continuous bilinear form (1) is coercive on $(W^1(\Omega, R^{\infty}))^n$, that is, there exists $\lambda > 0$, $\lambda \in R^1$ such that

$$\pi(t; \bar{y}, \bar{y}) \geqslant \lambda \|\bar{y}\|^2_{(W^1(\Omega,R^{\infty}))^n}, \tag{2}$$
$$\forall \bar{y}, \bar{\varphi} \in \left(W^1\left(\Omega, R^{\infty}\right)\right)^n \quad \text{the function } t \to \pi(t; \bar{y}, \bar{\varphi}) \text{ is measurable on } (0,T). \tag{3}$$

Under the above assumptions, in view of Kowalewski's results [13–16] and Lions and Magenes [18, vol. 2, Theorem 15.2, p. 8] with $f_i(x,t) = g_i(x,t) - b_i(x,t)y_i(x,t-h(t))$, $f_i(x,t) \in W^{-\frac{1}{2},-\frac{1}{4}}(Q)$ we can formulate the existence and uniqueness of solution of the following mixed initial boundary value problem for $n \times n$ parabolic lag system which defines the state of system model:

If (2) and (3) hold, then there exists a unique solution $y_i(\bar{v}) \in W^{\frac{3}{2},\frac{3}{4}}(Q)$ such that $\forall 1 \leqslant i \leqslant n$,

$$\left.\begin{aligned}
&\frac{\partial y_i(\bar{v})}{\partial t} + A(t)y_i(\bar{v}) + b_i(x,t)y_i\big(x,t-h(t);\bar{v}\big) = g_i, && \text{in } Q = \Omega \times (0,T),\\
&y_i(x,t') = \varphi_{i,0}(x,t'), && \text{in } Q_0 = \Omega \times \big[-h(0),0\big),\\
&y_i(x,0) = y_{i,0}(x), && \text{in } \Omega,\\
&\frac{\partial y_i(\bar{v})}{\partial \eta_A} = c_i(x,t)y_i\big(x,t-h(t);\bar{v}\big) + v_i, && \text{on } \Sigma = \Gamma \times (0,T),\\
&y_i(x,t') = \psi_{i,0}(x,t'), && \text{on } \Sigma_0 = \Gamma \times \big[-h(0),0\big),
\end{aligned}\right\} \tag{4}$$

where $y_{i,0} \in W^{\frac{1}{2}}(\Omega, R^{\infty})$, $\varphi_{i,0} \in W^{\frac{3}{2},\frac{3}{4}}(Q_0)$, $\psi_{i,0} \in L_2(\Sigma_0)$, $v_i \in L_2(\Sigma)$ and $g_i \in W^{-\frac{1}{2},-\frac{1}{4}}(Q)$ are given,

$$y_i(\bar{v}) \equiv y_i(x,t;\bar{v}), \qquad g_i \equiv g_i(x,t), \qquad v_i \equiv v_i(x,t),$$

b_i is a given real C^{∞} function defined on $\bar{Q} = \bar{\Omega} \times [0,T]$ ($\bar{Q}$ closure of Q), c_i is a given real C^{∞} function defined on Σ, $h(t)$ is a function representing a time-varying lag, $\varphi_{i,0}$ is an initial function defined on Q_0, $\psi_{i,0}$ is an initial function defined on Σ_0, the operator $(\frac{\partial}{\partial t} + A(t))$ is second order parabolic operator and $A(t)$ takes the form given above.

The $n \times n$ coupled system (4) constitutes a Neumann problem. Then the left-hand side of the Neumann boundary condition is written in the form

$$\frac{\partial y_i(\bar{v})}{\partial \eta_A} = \sum_{k=1}^{\infty} \big(D_k y_i(\bar{v})\big) \cos(n, x_k) = d_i(x,t) \in L_2(\Sigma),$$

where $\frac{\partial}{\partial \eta_A}$ is the normal derivative at Γ, directed towards the exterior of Ω, $\cos(n, x_k)$ is the kth direction cosine of n, n being the normal at Γ exterior to Ω and

$$d_i(x,t) = \big(c_i(x,t) y_i\big(x, t-h(t)\big) + v_i(x,t)\big) \in L_2(\Sigma).$$

For any pair of real numbers $r, s \geqslant 0$ we introduce the Sobolev space $W^{r,s}(Q)$ (Lions and Magenes [18, vol. 2, p. 6]) by

$$W^{r,s}(Q) = L_2\big(0, T; W^r\big(\Omega, R^\infty\big)\big) \cap W^s\big(0, T; L_2\big(\Omega, R^\infty\big)\big)$$

which is a Hilbert space normed by

$$\left(\int_0^T \|y_i(t)\|^2_{W^r(\Omega, R^\infty)}\, dt + \|y_i\|^2_{W^s(0,T;L_2(\Omega, R^\infty))} \right)^{\frac{1}{2}}.$$

Let $t - h(t)$ be strictly increasing function, $h(t)$ being non-negative in $[0, T]$ and also being a C^1 function. Then, there exists the inverse function of $t - h(t)$. Let us denote $r(t) \triangleq t - h(t)$, then the inverse function of $r(t)$ has the form $t = r + s(r)$, where $s(r)$ is time-varying prediction.

2. Linear quadratic boundary control problem

Let us denote by $U = (L_2(\Sigma))^n$ the space of controls. For the control $\bar{v} = (v_i)_{i=1}^n \in (L_2(\Sigma))^n$, the state of the system $y_i(\bar{v}) \in W^{\frac{3}{3},\frac{3}{4}}(Q) \subset L_2(Q)$ is given by the solution of (4). The time horizon T is fixed in our problem.

We observe $\bar{y}(\bar{v})$ in Q and for $z_{i,d} \in L_2(Q)$ the performance functional is given by

$$\begin{aligned} I(\bar{v}) &= I_1(\bar{v}) + I_2(\bar{v}) + \cdots + I_n(\bar{v}) \\ &= \sum_{i=1}^{n} \left[\lambda_1 \int_Q \big| y_i(x,t;\bar{v}) - z_{i,d} \big|^2 \, d\rho\, dt + \lambda_2 \int_\Sigma (N_i v_i) v_i \, d\Gamma\, dt \right], \end{aligned} \tag{5}$$

where $\lambda_i \geqslant 0$ and $\lambda_1 + \lambda_2 > 0$, $(N_i)_{i=1}^n$ is a diagonal matrix of strictly positive linear operator on $(L_2(\Sigma))^n$ into $(L_2(\Sigma))^n$.

Finally, we assume the following constraint on control $\bar{v} \in U_{\mathrm{ad}}$ (set of admissible controls), where

$$U_{\mathrm{ad}} \quad \text{is a closed, convex subset of } U. \tag{6}$$

For any control $\bar{v} \in U_{\mathrm{ad}}$, we note that the performance functional (5) is well defined since $y_i(\bar{v}) \in W^{\frac{3}{2},\frac{3}{4}}(Q) \subset L_2(Q)$.

The boundary control problem is to find

$$\inf_{\bar{v} \in U_{\mathrm{ad}}} I(\bar{v}).$$

Solving of this problem is equivalent to a seeking $\bar{v}^0 \in U_{\mathrm{ad}}$ such that

$$\sum_{i=1}^{n} I_i(\bar{v}^0) \leqslant \sum_{i=1}^{n} I_i(\bar{v}), \quad \forall \bar{v} \in U_{\mathrm{ad}},$$

which is given by the following theorem.

Theorem. *For the problem* (4) *with the performance functional* (5), $z_{i,d} \in L_2(Q)$ *and* $\lambda_2 > 0$ *and with constraints on controls* (6), *there exists the unique optimal boundary control* $\bar{v}^0$ *which satisfies the following maximum condition*:

$$\sum_{i=1}^{n} \int_{\Sigma} \left(P_i(\bar{v}^0) + \lambda_2 N_i v_i^0\right)\left(v_i - v_i^0\right) d\Gamma\, dt \geqslant 0, \quad \forall \bar{v} \in U_{\mathrm{ad}}.$$

Proof. From Theorem 1.3 of Lions [17, p. 10] it follows that for $\lambda_2 > 0$, the unique optimal control $\bar{v}^0$ exists, moreover $\bar{v}^0$ is characterized by

$$\sum_{i=1}^{n} I_i'(\bar{v}^0)\left(v_i - v_i^0\right) \geqslant 0, \quad \forall \bar{v} \in U_{\mathrm{ad}}.$$

Using the form of the performance functional (5), we can express the above inequality in the form

$$\sum_{i=1}^{n} \lambda_1 \int_{Q} \left[y_i(\bar{v}^0) - z_{i,d}\right]\left[y_i(\bar{v}) - y_i(\bar{v}^0)\right] d\rho\, dt + \sum_{i=1}^{n} \lambda_2 \int_{\Sigma} N_i v_i^0\left(v_i - v_i^0\right) d\Gamma\, dt \geqslant 0. \tag{7}$$

The above inequality can be simplified by introducing an $n \times n$ adjoint equation as

$$\left.\begin{array}{ll}
-\dfrac{\partial P_i(\bar{v})}{\partial t} + A^*(t)P_i(\bar{v}) + b_i\big(x, t+s(t)\big)P_i\big(x, t+s(t); \bar{v}\big)\big(1+s'(t)\big) & \\
\quad = \lambda_1\big(y_i(\bar{v}) - z_{i,d}\big), & \text{in } \Omega \times \big(0, T-h(0)\big), \\
-\dfrac{\partial P_i(\bar{v})}{\partial t} + A^*(t)P_i(\bar{v}) = \lambda_1\big(y_i(\bar{v}) - z_{i,d}\big), & \text{in } \Omega \times \big(T-h(T), T\big), \\
P(x, T; \bar{v}) = 0, & \text{in } x \in \Omega, \\
\dfrac{\partial P_i(\bar{v})}{\partial \eta_{A^*}} = \begin{cases} c_i(x, t+s(t))P_i(x, t+s(t); \bar{v})(1+s'(t)), \\ 0, \end{cases} & \begin{array}{l}\text{on } \Gamma \times (0, T-h(T)), \\ \text{on } \Gamma \times (T-h(T), T),\end{array}
\end{array}\right\} \tag{8}$$

where

$$\frac{\partial P_i(\bar{v})}{\partial \eta_{A^*}} = \sum_{k=1}^{\infty}\big(D_k P_i(\bar{v})\big)\cos(n, x_k),$$

$$A^*(t)P_i(\bar{v}) = \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t)\right)P_i(\bar{v}) + \sum_{j=1}^{n} a_{ij}P_j(\bar{v}),$$

a_{ji} is the transpose of a_{ij}.

As above in the previous section, for given $z_{i,d} \in L_2(Q)$ and any $\bar{v} \in L_2(\Sigma)$ there exists the unique solution $P_i(\bar{v}) \in W^{\frac{3}{2},\frac{4}{3}}(Q)$ for problem (8).

We simplify (7) using the $n \times n$ adjoint equation (8). For this purpose, setting $\bar{v} = \bar{v}^0$ in (8) and multiplying both sides of the first and the second equation of (8) by $[y_i(\bar{v}) - y_i(\bar{v}^0)]$, then integrating over $\Omega \times (0, T-h(T))$ and $\Omega \times (T-h(T), T)$ respectively, and then adding both sides we get

$$\begin{aligned}
&\lambda_1 \int_Q \big[y_i(\bar{v}^0) - z_{i,d}\big]\big[y_i(\bar{v}) - y_i(\bar{v}^0)\big]\, d\rho\, dt \\
&= \int_Q \bigg[-\frac{\partial P_i(\bar{v}^0)}{\partial t} + A^*(t) P_i(\bar{v}^0)\bigg]\big[y_i(\bar{v}) - y_i(\bar{v}^0)\big]\, d\rho\, dt \\
&\quad + \int_0^{T-h(T)} \int_\Omega b_i\big(x, t+s(t)\big) P_i\big(x, t+s(t); \bar{v}^0\big)\big(1+s'(t)\big) \\
&\quad \times \big[y_i(x,t;\bar{v}) - y_i\big(x,t;\bar{v}^0\big)\big]\, d\rho\, dt \\
&= \int_Q P_i(\bar{v}^0) \frac{\partial}{\partial t}\big[y_i(\bar{v}) - y_i(\bar{v}^0)\big]\, d\rho\, dt + \int_Q A^*(t) P_i(\bar{v}^0)\big[y_i(\bar{v}) - y_i(\bar{v}^0)\big]\, d\rho\, dt \\
&\quad + \int_0^{T-h(T)} \int_\Omega b_i\big(x, t+s(t)\big) P_i\big(x, t+s(t); \bar{v}^0\big)\big(1+s'(t)\big) \\
&\quad \times \big[y_i(x,t;\bar{v}) - y_i\big(x,t;\bar{v}^0\big)\big]\, d\rho\, dt.
\end{aligned}$$

Using Green's formula for the second term of the above right-hand side, we have

$$\begin{aligned}
&\lambda_1 \int_Q \big[y_i(\bar{v}^0) - z_{i,d}\big]\big[y_i(\bar{v}) - y_i(\bar{v}^0)\big]\, d\rho\, dt \\
&= \int_Q P_i(\bar{v}^0) \frac{\partial}{\partial t}\big[y_i(\bar{v}) - y_i(\bar{v}^0)\big]\, d\rho\, dt + \int_Q P_i(\bar{v}^0) A(t)\big[y_i(\bar{v}) - y_i(\bar{v}^0)\big]\, d\rho\, dt \\
&\quad + \int_0^T \int_\Gamma P_i(\bar{v}^0)\bigg[\frac{\partial y_i(\bar{v})}{\partial \eta_A} - \frac{\partial y_i(\bar{v}^0)}{\partial \eta_A}\bigg]\, d\Gamma\, dt - \int_0^T \int_\Gamma \frac{\partial y_i(\bar{v}^0)}{\partial \eta_{A^*}}\big[y_i(\bar{v}) - y_i(\bar{v}^0)\big]\, d\Gamma\, dt \\
&\quad - \int_0^{T-h(T)} \int_\Omega b_i\big(x, t+s(t)\big) P_i\big(x, t+s(t); \bar{v}^0\big)\big(1+s'(t)\big) \\
&\quad \times \big[y_i(x,t;\bar{v}) - y_i\big(x,t;\bar{v}^0\big)\big]\, d\rho\, dt. \qquad (9)
\end{aligned}$$

Using the Neumann boundary condition of (4), third term of the right-hand side of (9) can be expressed as

$$\begin{aligned}
&\int_0^T \int_\Gamma P_i(\bar{v}^0)\bigg[\frac{\partial y_i(\bar{v})}{\partial \eta_A} - \frac{\partial y_i(\bar{v}^0)}{\partial \eta_A}\bigg]\, d\Gamma\, dt \\
&= \int_0^T \int_\Gamma P_i\big(x,t;\bar{v}^0\big) c_i(x,t)\big[y_i\big(x, t-h(t); \bar{v}\big) - y_i\big(x, t-h(t); \bar{v}^0\big)\big]\, d\Gamma\, dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T\int_\Gamma P_i\big(x,t;\bar v^0\big)\big(v_i-v_i^0\big)\,d\Gamma\,dt\\
&=\int_{-h(0)}^{T-h(T)}\int_\Gamma P_i\big(x,t'+s(t');\bar v^0\big)c_i\big(x,t'+s(t')\big)\big(1+s'(t')\big)\\
&\quad\times\big[y_i(x,t';\bar v)-y_i\big(x,t';\bar v^0\big)\big]\,d\Gamma\,dt'\\
&\quad+\int_0^T\int_\Gamma P_i\big(x,t;\bar v^0\big)\big(v_i-v_i^0\big)\,d\Gamma\,dt\\
&=\int_{-h(0)}^{0}\int_\Gamma P_i\big(x,t'+s(t');\bar v^0\big)c_i\big(x,t'+s(t')\big)\big(1+s'(t')\big)\\
&\quad\times\big[y_i(x,t';\bar v)-y_i\big(x,t';\bar v^0\big)\big]\,d\Gamma\,dt'\\
&\quad+\int_{0}^{T-h(T)}\int_\Gamma P_i\big(x,t'+s(t');\bar v^0\big)c_i\big(x,t'+s(t')\big)\big(1+s'(t')\big)\\
&\quad\times\big[y_i(x,t';\bar v)-y_i\big(x,t';\bar v^0\big)\big]\,d\Gamma\,dt'\\
&\quad+\int_0^T\int_\Gamma P_i\big(x,t;\bar v^0\big)\big(v_i-v_i^0\big)\,d\Gamma\,dt.
\end{aligned}
\tag{10}
$$

Also, using the Neumann boundary condition of (8), the fourth term in (9) can be rewritten as

$$
\begin{aligned}
&\int_0^T\int_\Gamma \frac{\partial P_i(\bar v^0)}{\partial\eta_{A^*}}\big[y_i(\bar v)-y_i\big(\bar v^0\big)\big]\,d\Gamma\,dt\\
&\quad=\int_0^{T-h(T)}\int_\Gamma \frac{\partial P_i(\bar v^0)}{\partial\eta_{A^*}}\big[y_i(\bar v)-y_i\big(\bar v^0\big)\big]\,d\Gamma\,dt\\
&\quad\quad+\int_{T-h(T)}^{T}\int_\Gamma \frac{\partial P_i(\bar v^0)}{\partial\eta_{A^*}}\big[y_i(\bar v)-y_i\big(\bar v^0\big)\big]\,d\Gamma\,dt\\
&\quad=\int_0^{T-h(T)}\int_\Gamma c_i\big(x,t+s(t)\big)P_i\big(x,t+s(t);\bar v^0\big)\big(1+s'(t)\big)\\
&\quad\quad\times\big[y_i(x,t;\bar v)-y_i\big(x,t;\bar v^0\big)\big]\,d\Gamma\,dt.
\end{aligned}
\tag{11}
$$

Substituting (10), (11) into (9) we obtain

$$
\lambda_1\int_Q\big[y_i\big(\bar v^0\big)-z_{i,d}\big]\big[y_i(\bar v)-y_i\big(\bar v^0\big)\big]\,dx\,dt
$$

$$
\begin{aligned}
&= \int\limits_Q P_i(\bar v^0)\left(\frac{\partial}{\partial t} + A(t)\right)\left[y_i(\bar v) - y_i(\bar v^0)\right] d\rho\, dt \\
&\quad + \int\limits_{-h(0)}^{0} \int\limits_\Gamma P_i\big(x, t + s(t); \bar v^0\big) c_i\big(x, t + s(t)\big)\big(1 + s'(t)\big) \\
&\quad \times \left[y_i(x, t; \bar v) - y_i\big(x, t; \bar v^0\big)\right] d\Gamma\, dt \\
&\quad + \int\limits_{0}^{T-h(T)} \int\limits_\Gamma P_i\big(x, t + s(t); \bar v^0\big) c_i\big(x, t + s(t)\big)\big(1 + s'(t)\big) \\
&\quad \times \left[y_i(x, t; \bar v) - y_i\big(x, t; \bar v^0\big)\right] d\Gamma\, dt \\
&\quad + \int\limits_0^T \int\limits_\Gamma P_i\big(x, t; \bar v^0\big)\big(v_i - v_i^0\big)\, d\Gamma\, dt \\
&\quad \times \int\limits_{0}^{T-h(T)} \int\limits_\Gamma c_i\big(x, t + s(t)\big) P_i\big(x, t + s(t); \bar v^0\big)\big(1 + s'(t)\big) \\
&\quad \times \left[y_i(x, t; \bar v) - y_i\big(x, t; \bar v^0\big)\right] d\Gamma\, dt \\
&\quad \times \int\limits_{0}^{T-h(T)} \int\limits_\Omega b_i\big(x, t + s(t)\big) P_i\big(x, t + s(t); \bar v^0\big)\big(1 + s'(t)\big) \\
&\quad \times \left[y_i(x, t; \bar v) - y_i\big(x, t; \bar v^0\big)\right] d\rho\, dt \\
&\quad + \int\limits_0^T \int\limits_\Gamma P_i\big(x, t; \bar v^0\big)\big(v_i - v_i^0\big)\, d\Gamma\, dt.
\end{aligned}
\tag{12}
$$

Further substituting (12) into (7) gives

$$
\sum_{i=1}^{n} \int\limits_\Sigma \big(P_i(\bar v^0) + \lambda_2 N_i v_i\big)\big(v_i - v_i^0\big)\, d\Gamma\, dt \geqslant 0
$$

which completes the proof. □

Note 1. We can also consider an analogous optimal boundary control problem where the performance functional is given by

$$
I(\bar v) = \sum_{i=1}^{n}\left[\lambda_1 \int\limits_\Sigma \left|y_i(\bar v) - z_{i,d}\right|^2 d\Gamma\, dt + \lambda_2 \int\limits_\Sigma (N_i v_i) v_i\, d\Gamma\, dt\right].
$$

From the above section and the trace theorem (Lions and Magenes [18, vol. 2, p. 9]) for each $\bar v \in (L_2(\Sigma))^n$, there exists a unique solution $y_i(\bar v) \in W^{\frac{3}{2},\frac{3}{4}}(Q)$ of the problem (4) with $y_i(\bar v)|_\Sigma \in W^{1,\frac{1}{2}}(\Sigma) \subset L_2(\Sigma)$. Let $z_{i,d} \in L_2(\Sigma)$, thus the above performance functional is well defined.

Then the optimal boundary control $\bar v^0$ is characterized by

$$\sum_{i=1}^{n}\lambda_1\int\limits_{\Sigma}\big(y_i(\bar{v}^0)-z_{i,d}\big)\big(y_i(\bar{v})-y_i(\bar{v}^0)\big)\,d\Gamma\,dt+\sum_{i=1}^{n}\lambda_2\int\limits_{\Sigma}\big(N_i v_i^0\big)\big(v_i-v_i^0\big)\,d\Gamma\,dt\geqslant 0. \tag{13}$$

We introduce the following $n\times n$ adjoint equation:

$$\begin{aligned}
&-\frac{\partial P_i(\bar{v}^0)}{\partial t}+A^*(t)P_i\big(\bar{v}^0\big) && \\
&\quad+b_i\big(x,t+s(t)\big)P_i\big(x,t+s(t);\bar{v}^0\big)\big(1+s'(t)\big)=0, && \text{in } \Omega\times\big(0,T-h(T)\big),\\
&-\frac{\partial P_i(\bar{v}^0)}{\partial t}+A^*(t)P_i\big(\bar{v}^0\big)=0, && \text{in } \Omega\times\big(T-h(T),T\big),\\
&P_i\big(x,T;\bar{v}^0\big)=0, && \text{in } \Omega,\\
&-\frac{\partial P_i(\bar{v}^0)}{\partial\eta_{A^*}}=\begin{cases} c_i(x,t+s(t))P_i(x,t+s(t);\bar{v}^0)(1+s'(t)) \\ \quad+\lambda_1(y_i(\bar{v}^0)-z_{i,d}), & \text{on } \Gamma\times(0,T-h(T)),\\ \lambda_1(y_i(\bar{v}^0)-z_{i,d}), & \text{on } \Gamma\times(T-h(T),T).\end{cases}
\end{aligned}$$

As above in the previous section, for given $z_{i,d}\in L_2(\Sigma)$ and any $\bar{v}^0\in(L_2(\Sigma))^n$ there exists a unique solution $P_i(\bar{v}^0)\in W^{\frac{3}{2},\frac{3}{4}}(Q)$ to the above problem.

In this case the condition (13) can be also rewritten in the following form:

$$\sum_{i=1}^{n}\int\limits_{\Sigma}\big(P_i\big(\bar{v}^0\big)+\lambda_2 N_i v_i^0\big)\big(v_i-v_i^0\big)\,d\Gamma\,dt\geqslant 0,\quad \forall\bar{v}\in U_{\mathrm{ad}}.$$

3. Mathematical examples

Example 1. If $U_{\mathrm{ad}}=U=(L_2(\Sigma))^n$, the case where there are no constraints on the controls. Thus the maximum condition is satisfied when

$$v_i^0=-\lambda_2^{-1}N_i^{-1}P_i\big(\bar{v}^0\big).$$

Example 2. If we consider the $n\times n$ parabolic coupled system in which a time-varying lag appears in the Neumann condition only, then the $n\times n$ system (4) becomes $\forall 1\leqslant i\leqslant n$,

$$\begin{aligned}
&\frac{\partial y_i(\bar{v})}{\partial t}+Ay_i(\bar{v})=g_i, && \text{in } Q=\Omega\times(0,T),\\
&y_i(x,0)=y_{i,0}(x), && \text{in } \Omega,\\
&\frac{\partial y_i(\bar{v})}{\partial\eta_A}=c_i(x,t)y_i\big(x,t-h(t)\big)+\bar{v}+v_i, && \text{in } \Sigma=\Gamma\times(0,T),\\
&y_i(x,t')=\psi_{i,0}(x,t'), && \text{in } \Sigma_0=\Gamma\times\big[-h(0),0\big).
\end{aligned}$$

The $n\times n$ adjoint equations (8) take the form

$$\begin{aligned}
&-\frac{\partial P_i(\bar{v})}{\partial t}+A^*P_i(\bar{v})=\lambda_1\big[y_i(\bar{v})-z_{i,d}\big], && \text{in } Q=\Omega\times(0,T),\\
&P(x,T;\bar{v})=0, && \text{in } \Omega,\\
&\frac{\partial P_i(\bar{v})}{\partial\eta_{A^*}}=\begin{cases} c_i(x,t+s(t))P_i(x,t+s(t);\bar{v})(1+s'(t)), & \text{on } \Gamma\times\big(0,T-h(T)\big),\\ 0, & \text{on } \Gamma\times\big(T-h(T),T\big).\end{cases}
\end{aligned}$$

Finally the maximum condition is

$$\sum_{i=1}^{n}\int\limits_{\Sigma}\big(P_i\big(\bar{v}^0\big)+\lambda_2 N_i v_i^0\big)\big(v_i-v_i^0\big)\,d\Gamma\,dt\geqslant 0.$$

Example 3. If we consider the $n \times n$ parabolic coupled system in which a time-varying lag appear in the equation only, then the $n \times n$ system (4) becomes $\forall 1 \leqslant i \leqslant n$,

$$\begin{aligned}
&\frac{\partial y_i(\bar v)}{\partial t} + Ay_i(\bar v) + b_i(x,t)y_i\big(x,t-h(t);\bar v\big) = g_i, && \text{in } Q,\\
&y_1(x,t') = \varphi_{i,0}(x,t'), && \text{in } Q_0,\\
&y_i(x,0) = y_{i,0}(x), && \text{in } \Omega,\\
&\frac{\partial y_i(\bar v)}{\partial \eta_A} = 0, && \text{on } \Sigma.
\end{aligned}$$

The $n \times n$ adjoint equation (8) takes the form

$$\begin{aligned}
&-\frac{\partial P_i(\bar v)}{\partial t} + A^*(t)P_i(\bar v)\\
&\qquad + b_i\big(x,t+h(t)\big)P_i\big(x,t+h(t);\bar v\big)\big(1+s'(t)\big)\\
&\quad = \lambda_1\big(y_i(\bar v) - z_{i,d}\big), && \text{in } \Omega \times \big(0, T-h(T)\big),\\
&-\frac{\partial P_i(\bar u)}{\partial t} + A^*P_i(\bar v) = \lambda_1\big(y_i(\bar v) - z_{i,d}\big), && \text{in } \Omega \times \big(T-h(T), T\big),\\
&P_i(x,T;\bar v) = 0, && \text{in } \Omega,\\
&\frac{\partial P_i(\bar v)}{\partial \eta_{A^*}} = 0, && \text{on } \Sigma.
\end{aligned}$$

Finally the maximum condition is

$$\sum_{i=1}^{n} \int_Q \big(P_i\big(\bar v^0\big) + \lambda_2 N_i v_i^0\big)\big(v_i - v_i^0\big)\, d\Gamma\, dt \geqslant 0.$$

Example 4. If $h(t) = h =$ constant, i.e., system with constant time lag [7], then the $n \times n$ system (4) becomes $\forall 1 \leqslant i \leqslant n$,

$$\begin{aligned}
&\frac{\partial y_i(\bar v)}{\partial t} + A(t)y_i(\bar v) + b_i(x,t)y_i(x,t-h;\bar v) = g_i, && \text{in } Q = \Omega \times (0,T),\\
&y_i(x,t') = \varphi_{i,0}(x,t'), && \text{in } Q_0 = \Omega \times (-h,0),\\
&y_i(x,0) = y_{i,0}(x), && \text{in } \Omega,\\
&\frac{\partial y_i(\bar v)}{\partial \eta_A} = c_i(x,t)y_i(x,t-h;\bar v) + v_1, && \text{on } \Sigma = \Gamma \times (0,T),\\
&y_i(x,t') = \psi_{i,0}(x,t'), && \text{on } \Sigma_0 = \Gamma \times (-h,0).
\end{aligned}$$

The $n \times n$ adjoint system (8) takes the form

$$\begin{aligned}
&-\frac{\partial P_i(\bar v)}{\partial t} A^*(t)P_i(\bar v) + b_i(x,t+h)P_i(x,t+h;\bar v)\\
&\quad = \lambda_1\big(y_i(\bar v) - z_{i,d}\big), && \text{in } \Omega \times (0,T-h),\\
&-\frac{\partial P_i(\bar v)}{\partial t} + A^*(t)P_i(\bar v) = \lambda_1\big(y_i(\bar v) - z_{i,d}\big), && \text{in } \Omega \times (T-h,T),\\
&P_i(x,T;\bar v) = 0, && \text{in } \Omega,\\
&\frac{\partial P_i(\bar v)}{\partial \eta_{A^*}} = \begin{cases} c_i(x,t+h)P_i(x,t+h;\bar v), & \text{on } \Gamma \times (0,T-h),\\ 0, & \text{on } \Gamma \times (T-h,T). \end{cases}
\end{aligned}$$

Finally the maximum condition is

$$\sum_{i=1}^{n} \int_\Sigma \big(P_i\big(\bar v^0\big) + \lambda_2 N_i v_i^0\big)\big(v_i - u_i^0\big)\, d\Gamma\, dt \geqslant 0.$$

Note 2. As in Examples 2 and 3 we can construct the problem of $n \times n$ parabolic system with constant time lag appearing in the Neumann boundary condition only, and the problem for $n \times n$ parabolic system with constant time lag appearing in the equation only, respectively.

Example 5. If we take $n = 2$ then $U = L_2(Q) \times L_2(Q)$, and the optimality system is given by

$$
\begin{aligned}
&\frac{\partial y_1(\bar v)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1\right) y_1(\bar v) - y_2(\bar v) \\
&\quad + b_1(x,t) y_1\big(x, t-h(t); \bar v\big) = v_1, \quad \text{in } Q, \\
&\frac{\partial y_2(\bar v)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1\right) y_2(\bar v) + y_1(\bar v) \\
&\quad + b_2(x,t) y_2\big(x, t-h(t); \bar v\big) = v_2, \quad \text{in } Q, \\
&y_1(x,t') = \varphi_{1,0}(x), \quad y_2(x,t') = \varphi_{2,0}(x), \quad \text{in } Q_0, \\
&y_1(x,0) = y_{1,0}(x), \quad y_2(x,0) = y_{2,0}(x), \quad \text{in } \Omega, \\
&\frac{\partial y_1(\bar v)}{\partial \eta_A} = c_1(x,t) y_1\big(x, t-h(t); \bar v + v_1\big), \\
&\frac{\partial y_2(\bar v)}{\partial \eta_A} = c_2(x,t) y_2\big(x, t-h(t); \bar v + v_2\big), \quad \text{on } \Sigma, \\
&y_1(x,t') = \psi_{1,0}(x), \quad y_2(x,t') = \psi_{2,0}(x), \quad \text{on } \Sigma_0,
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\partial P_1(\bar v)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1\right) P_1(\bar v) + P_2(\bar v) + b_1\big(x, t+s(t)\big) P_1\big(x, t+s(t); \bar v\big) \\
&\quad \times \big(1 + s'(t')\big) = \lambda_1\big(y_1(\bar v) - z_{1,d}\big), \quad \text{in } \Omega \times \big(0, T-h(T)\big), \\
&-\frac{\partial P_1(\bar v)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1\right) P_1(\bar v) + P_2(\bar v) \\
&\quad = \lambda_1\big(y_1(\bar v) - z_{1,d}\big), \quad \text{in } \Omega \times \big(T-h(T), T\big), \\
&-\frac{\partial P_2(\bar v)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1\right) P_2(\bar v) - P_1(\bar v) + b_2\big(x, t+s(t)\big) P_2\big(x, t+s(t); \bar v\big) \\
&\quad \times \big(1 + s'(t')\big) = \lambda_1\big(y_2(\bar v) - z_{2,d}\big), \quad \text{in } \Omega \times \big(0, T-h(T)\big), \\
&-\frac{\partial P_2(\bar v)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1\right) P_2(\bar v) - P_1(\bar v) \\
&\quad = \lambda_2\big(y_2(\bar v) - z_{2,d}\big), \quad \text{in } \Omega \times \big(T-h(T), T\big), \\
&P_1(x,T;\bar v) = 0, \quad P_2(x,T;\bar v) = 0, \quad \text{in } \Omega, \\
&\frac{\partial P_1(\bar v)}{\partial \eta_{A^*}} = \begin{cases} c_1(x, t+s(t)) P_1(x, t+s(t), \bar v)(1 + s'(t)), & \text{on } \Gamma \times (0, T-h(T)), \\ 0, & \text{on } \Gamma \times (T-h(T), T), \end{cases} \\
&\frac{\partial P_2(\bar v)}{\partial \eta_{A^*}} = \begin{cases} c_2(x, t+s(t)) P_2(x, t+s(t); \bar v)(1 + s'(t)), & \text{on } \Gamma \times (0, T-h(T)), \\ 0, & \text{on } \Gamma \times (T-h(T), T), \end{cases}
\end{aligned}
$$

$$\int_Q \left[\left(P_1(\bar{v}^0) + \lambda_2 N_1 v_1^0\right)\left(v_1 - v_1^0\right) + \left(P_2(\bar{v}^0) + \lambda_2 N_2 v_2^0\right)\left(v_2 - v_2^0\right)\right] d\rho\, dt \geqslant 0$$
$$\forall (v_1, v_2) \in U_{\mathrm{ad}}, \ \left(v_1^0, v_2^0\right) \in U_{\mathrm{ad}}.$$

Example 6. If $n = 2$ and $U_{\mathrm{ad}} = U$, then the optimal control $\bar{v}^0 = (v_1^0, v_2^0)$ is obtained by solving the following system of partial differential equations:

$$\left.\begin{aligned}
&\frac{\partial y_1(\bar{v}^0)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1\right) y_1(\bar{v}^0) - y_2(\bar{v}^0) \\
&\quad + b_1(x,t) y_1\left(x, t - h(t); \bar{v}^0\right) - \lambda_2^{-1} N_1^{-1} P_1(\bar{v}^0) = 0, \\
&\frac{\partial y_2(\bar{v}^0)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1\right) y_2(\bar{v}^0) + y_1(\bar{v}^0) \\
&\quad + b_2(x,t) y_2\left(x, t - h(t); \bar{v}^0\right) - \lambda_2^{-1} N_2^{-1} P_2(\bar{v}^0) = 0,
\end{aligned}\right\} \quad \text{in } Q,$$

$$y_1(x, t') = \varphi_{1,0}(x), \quad y_2(x, t') = \varphi_{2,0}(x), \quad \text{in } Q_0,$$

$$-\frac{\partial P_1(\bar{v}^0)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1\right) P_1(\bar{v}^0) + P_2(\bar{v}^0) + b_1\left(x, t + s(t)\right)$$
$$\times P_1\left(x, t + s(t); \bar{v}^0\right)\left(1 + s'(t)\right) = \lambda_1\left(y_1(\bar{v}^0) - z_{1,d}\right), \quad \text{in } \Omega \times \left(0, T - h(T)\right),$$

$$-\frac{\partial P_2(\bar{v}^0)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2 + q(x,t) + 1\right) P_2(\bar{v}^0) - P_1(\bar{v}^0) + b_2\left(x, t + s(t)\right)$$
$$\times P_2\left(x, t + s(t); \bar{v}^0\right)\left(1 + s'(t')\right) = \lambda_2\left(y_2(\bar{v}^0) - z_{2,d}\right), \quad \text{in } \Omega \times \left(0, T - h(T)\right),$$

$$\left.\begin{aligned}
&y_1(x,0) = y_{1,0}(x), \quad y_2(x,0) = y_{2,0}(x) \\
&P_1(x,T;\bar{v}^0) = 0, \qquad P_2(x,T;\bar{v}^0) = 0
\end{aligned}\right\}, \quad \text{in } \Omega,$$

$$\frac{\partial y_1(\bar{v}^0)}{\partial t} = c_1(x,t) y_1\left(x, t - h(t); \bar{v}^0\right) + g_1,$$
$$\frac{\partial y_2(\bar{v}^0)}{\partial t} = c_2(x,t) y_2\left(x, t - h(t); \bar{v}^0\right) + g_2, \quad \text{on } \Sigma,$$
$$y_1(x, t') = \psi_{1,0}(x), \quad y_2(x, t') = \psi_{2,0}(x), \quad \text{on } \Sigma_0,$$
$$\frac{\partial P_1(\bar{v}^0)}{\partial \eta_{A^*}} = \begin{cases} c_1(x, t + s(t)) P_1(x, t + s(t); \bar{v}^0)(1 + s'(t)), & \text{on } \Gamma \times (0, T - h(T)), \\ 0, & \text{on } \Gamma \times (T - h(T), T), \end{cases}$$
$$\frac{\partial P_2(\bar{v}^0)}{\partial \eta_{A^*}} = \begin{cases} c_2(x, t + s(t)) P_2(x, t + s(t); \bar{v}^0)(1 + s'(t)), & \text{on } \Gamma \times (0, T - h(T)), \\ 0, & \text{on } \Gamma \times (T - h(T), T). \end{cases}$$

Further

$$v_1^0 = -\lambda_2^{-1} N_1^{-1} P_1(\bar{v}^0), \qquad v_2^0 = -\lambda_2^{-1} N_2^{-1} P_2(\bar{v}^0).$$

Note 3. We observe that the conditions of optimality derived above allow us to obtain an analytical formula for the optimal control in particular case only (i.e., where there are no constraints on controls). This results from the following: the determining of the function $P_i(\bar{u}^0)$ in the maximum condition from the adjoint equation is possible if and only if we know $\bar{y}^0$ which corresponds to the control $\bar{u}^0$. These mutual connections make the practical use of the derived optimization formulas difficult. Therefore we must resign from the exact determining of the optimal control and we are forced to use approximation methods.

In the case of performance functional with $\lambda_1 > 0$ and $\lambda_2 = 0$, the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadratic programing which can be solved by the use of the well-known algorithms [13–16].

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